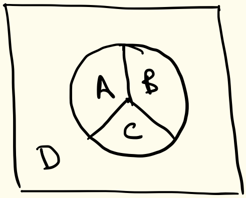
Kitaev-Preskill's Topological Entanglement Entropy



We are given a ground state wave-fn $|\psi\rangle$ of a gapped Hamiltonian.

Consider the combination

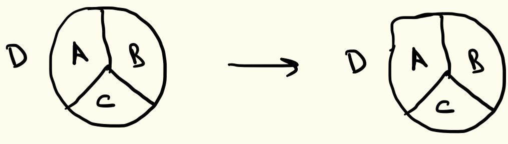
$$S_{topo} = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{CA} + S_{ABC}.$$

$A \cup B \cup C \cup D$ = whole system.

where S_X denotes von Neumann entropy of density matrix $\rho_X = \mathrm{Tr}_{\bar{X}} |\psi\rangle\langle\psi|$ and $AB = A \cup B$ etc.

All regions A, AB etc. are much larger than the correlation length associated with local operators.

This combination of entropies is 'topological' i.e. it is robust to perturbations of the Hamiltonian (assuming gap doesn't close) or small changes in the geometry of regions A, B, C. e.g. consider deforming region A a bit

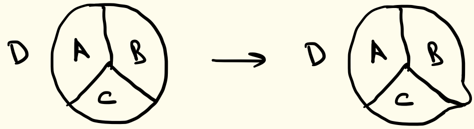


. The change in S_{topo} is

$$\Delta S_{topo} = \Delta S_A - \Delta S_{AB} - \Delta S_{CA} + \Delta S_{ABC}$$
$$= \Delta [S_A - S_{CA}] - \Delta [S_{AB} - S_{ABC}]$$

$S_A - S_{CA}$ is unchanged since it corresponds to change in entropy associated with attaching C to A, and boundary of C is far from the location where A is deformed. $\Rightarrow$ $\Delta[S_A - S_{AC}] \simeq 0$. Similarly $\Delta[S_{AB} - S_{ABC}] \simeq 0$ $\Rightarrow$ $\Delta S_{topo} \simeq 0$.

Similarly, consider deforming the geometry where three regions meet.



Now $\Delta S_{topo} = \Delta S_B + \Delta S_C - \Delta S_{AB} - \Delta S_{BC} - \Delta S_{CA} + \Delta S_{ABC}$

$$= \Delta[S_B - S_{AB}] + \Delta[S_{ABC} - S_{BC}] + \Delta[S_{ABD} - S_{BD}]$$

where we have used $S_X = S_{\bar{X}}$ for a pure state $\Rightarrow$ $S_C = S_{ABD}$ and $S_{CA} = S_{BD}$.

By the same argument as above, $\Delta S_{topo} \simeq 0$, since all three terms correspond to attaching region A to some region and region A B far from the location of geometry change.

If S_x for any spatial region x takes the form

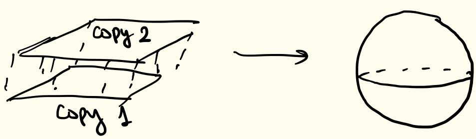
$$S_x = |\partial x| - \gamma,$$ we expect γ to be a universal number independent of the shape of region x, as long as x is finite in extent. If so,

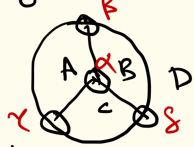
$$S_{topo} = |\partial A| - \gamma + |\partial B| - \gamma + |\partial C| - \gamma$$
$$- |\partial(AB)| + \gamma - |\partial(BC)| + \gamma - |\partial(CA)| + \gamma$$
$$+ |\partial(ABC)| - \gamma$$
$$= |\partial A| + |\partial B| + |\partial C| - [|\partial A| + |\partial B| - 2|\partial(A \cap B)|]$$
$$- [|\partial B| + |\partial C| - 2|\partial(B \cap C)|] - [|\partial C| + |\partial A| - 2|\partial(A \cap C)|]$$
$$+ |\partial A| + |\partial B| + |\partial C| - 2|\partial(A \cap B)| - 2|\partial(B \cap C)|$$
$$- 2|\partial(C \cap A)| - \gamma$$

Everything cancels out except $-\gamma$.

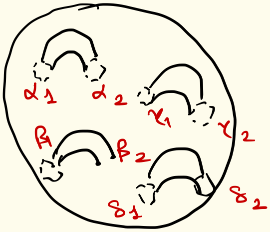
$$\Rightarrow \boxed{S_{topo} = -\gamma}$$

To calculate S_{topo}, we consider two copies to the system and glue them at spatial infinity, thus obtaining a sphere:



Next we attach wormholes that join the two copies at the intersection of any three regions. There are four such intersections: 

. The resulting topology is a sphere with four handles:



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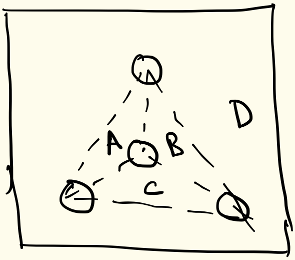


α_1, α_2 denote the mouth of the wormhole in two copies 1 and 2.

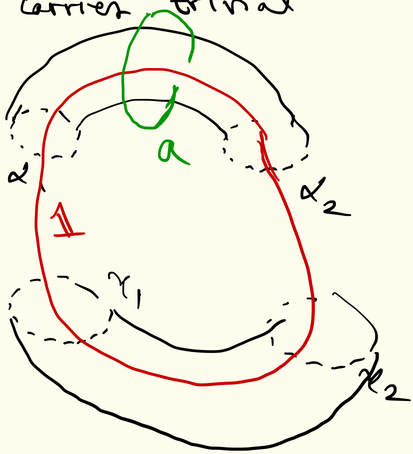
Now,

$$2S_{topo} = \underbrace{4S_3}_{(S_A+S_B+S_C+S_{ABC})} - \underbrace{3S_4}_{(S_{AB}+S_{BC}+S_{CA})}$$

where $S_A = S_B = S_C = S_{ABC} = S_3$
= entropy of a region with three punctures

$S_{AB} = S_{BC} = S_{CA} = S_4$
= entropy of a region with four punctures.

The mouth of each wormhole carries trivial (i.e. identity) anyon charge. e.g., the loop on the right should detect trivial anyon charge. This allows one to calculate the probability that the loop complimentary

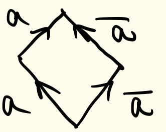


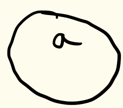
to it, i.e. a loop that has non-zero winding with the above loop carries charge of a specific anyon, say, a. We denote this complimentary loop for the wormhole associated with (α_1, α_2) the green loop above.

propability of anyon 'a' charge at puncture $(\alpha_1, \alpha_2) = |S_{1a}|^2$ where S is the S-matrix. This is because S_{ab} is the amplitude for braiding world-lines of anyon types a and b. Here $b = 1$. Using

$$S_{1a} = \frac{d_a}{D} \quad \text{where } D = \sqrt{\sum_i d_i^2},$$

$$P_a = \frac{d_a^2}{D^2}.$$

Heuristically, we are calculating the unconditional propability of finding an anyon of type 'a' at the puncture. This has the amplitude 

$=$ 

$= d_a$, from our earlier discussion of anyon algebra.

To calculate S_3, we need to calculate the probability of anyon charges a, b, c at the three punctures. These anyons must fuse to identity, again because the loop that winds around the throat of wormhole must detect $\mathbb{1}$. Denoting this probability as P_{abc},

$$P_{abc} = P_a P_b \, P_{ab \to \bar{c}} \quad \text{where}$$

$P_{ab \to \bar{c}}$ is the prob. that a, b fuse to $\bar{c}$ ($=$ prob. that a, b, c fuse to $\mathbb{1}$). To calculate $P_{ab \to \bar{c}}$, we note,

$$\sum_{ab} P_a \, P_b \, P_{ab \to \bar{c}} = P_c$$

where P_x are the unconditional prob. of finding x. Using $P_a = \frac{d_a^2}{D^2}$ and $d_a d_b = \sum_c N^c_{ab} d_c$,

One can check that the above eqn is satisfied if we chose,

$$P_{ab \to \bar{c}} = \frac{N^{\bar{c}}_{ab} \, d_c}{d_a d_b}$$

Check: $\displaystyle \sum_{ab} \frac{d_a^2 \, d_b^2}{D^4} \frac{N^{\bar{c}}_{ab} \, d_c}{d_a d_b} = \frac{\sum_b d_c^2 \, d_b^2}{D^4} = \frac{d_c^2}{D^2} = P_c$

$$\Rightarrow \quad P_{abc} = p_a p_b P_{ab\to\bar{c}}$$

$$= \frac{d_a^2}{D^2}\,\frac{d_b^2}{D^2}\,\frac{N^{\bar{c}}_{ab}\, d_c}{d_a d_b}$$

$$= \frac{d_a d_b d_c N_{abc}}{D^4}$$

Thus, $$S_3 = -\sum_{abc} P_{abc} \log P_{abc}$$

$$= -\sum_{abc} \frac{d_a d_b d_c\, N_{abc}}{D^4} \log\left[\frac{d_a d_b d_c}{D^4}\right]$$

$$= -\sum_{abc} \frac{d_a d_b d_c N_{abc}}{D^4}\left[\log(d_a d_b d_c) - 4\log D\right]$$

$$= 4\log D \sum_{ab} \frac{d_a^2\, d_b^2}{D^4}$$

$$-3\sum_{abc} \frac{d_a\, d_b\, d_c\, N_{abc}}{D^4} \log(d_a)$$

$$= 4\log D - 3\sum_{ab} \frac{d_a^2\, d_b^2}{D^4}\log(d_a)$$

$$= 4\log D - 3\sum p_a \log(d_a)$$

where $p_a = \frac{d_a^2}{D^2}$.

Similarly, $S_4 = 6 \log D - 4 \sum_a p_a \log d_a$

The terms proportional to $\sum_a p_a \log d_a$ are analog of the area law terms in $S \sim L - \gamma$ and cancel out:

$$2 S_{topo} = 4 S_3 - 3 S_4$$
$$= 4 \left[4 \log D - 3 \sum_a p_a \log d_a \right]$$
$$- 3 \left[6 \log D - 4 \sum_a p_a \log d_a \right]$$
$$= -2 \log D$$
$$\Rightarrow \quad S_{topo} = - \log D$$
$$\Rightarrow \quad \boxed{\gamma = \log D}$$

Examples: toric code $D = \sqrt{\underset{\mathbb{1}}{1^2} + \underset{e}{1^2} + \underset{m}{1^2} + \underset{em}{1^2}}$

$= 2$

$\gamma = \log(2)$.

Ising anyon: $D = \sqrt{\underset{\mathbb{1}}{1^2} + \underset{\psi}{1^2} + \underset{\sigma}{(\sqrt{2})^2}}$

$= \sqrt{4} = 2$

Fibonacci $D = \sqrt{1^2 + \varphi^2} = \sqrt{1 + \varphi^2}$ $\quad \varphi = \frac{1+\sqrt{5}}{2}$.